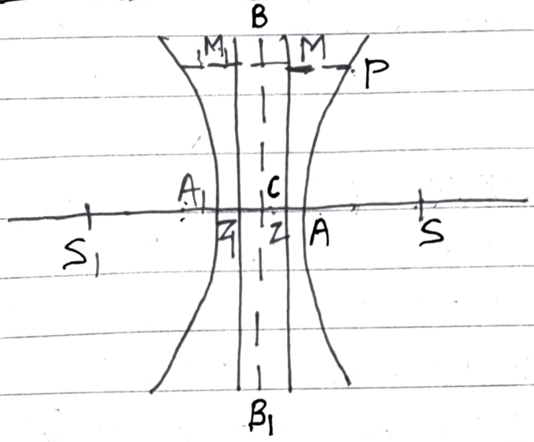
Hyperbola

It is the locus of a point which moves such that its distance from a fixed point is of a constant ratio (>1) to its distance from the fixed point.

The fixed point is called focus, the fixed line is called directrix and the ratio is called eccentricity. where $e > 1$

Equation of Hyperbola



let S be the focus and ZM the directrix. From S draw $SZ \perp ZM$ and divide internally at A & externally at A_1 in the ratio $e:1$

$$\Rightarrow \frac{AS}{AZ} = \frac{A_1S}{A_1Z} = \frac{e}{1}$$

Now, since A and A_1 are points on the hyperbola. Let $AA_1 = 2a$. and 'C' is the middle of AA_1

$$AS = e\,AZ \Rightarrow A_1S = e\,A_1Z$$

$$\Rightarrow AS + A_1S = e(AZ + A_1Z)$$

$$\Rightarrow (CS - \cancel{CA}) + (\cancel{A_1C} + CS) = e(CA - CZ + A_1C + CZ)$$

$$\Rightarrow 2CS = e\,A_1A \Rightarrow 2CS = 2ae$$

Also $A_1S - AS = e(A_1Z - AZ)$

$\Rightarrow A_1A = e(A_1C + CZ - AC - CZ)$

$\Rightarrow A_1A = e(2CZ)$

$\Rightarrow \cancel{2}a = \cancel{2}e\,CZ$

$\Rightarrow CZ = a/e$

x-axis

Now take 'C' as the origin & CAX as the axis and a line CY ⊥ CAS.

y-axis

∴ the co-ordinates of S are $(ae, 0)$

& equation of directrix is $x = a/e$

Let $P(x, y)$ be any point on hyperbola.

Draw $PM \perp ZM$. By Defination.

We know that $SP = ePM$

$\Rightarrow SP^2 = e^2 PM^2 = e^2(x - CZ)^2$

$\Rightarrow (x - ae)^2 + y^2 = e^2(x - a/e)^2$

$\Rightarrow x^2 + y^2 + a^2e^2 - 2xae = e^2\left(x^2 + \frac{a^2}{e^2} - 2x\frac{a}{e}\right)$

~~$\Rightarrow x^2(1-e^2)$~~

$\Rightarrow x^2 + y^2 + a^2e^2 - 2\cancel{x}ae = e^2x^2 + a^2 - 2\cancel{x}ae$

$\Rightarrow x^2(1 - e^2) + y^2 = a^2 - a^2e^2$

$\Rightarrow x^2(1 - e^2) + y^2 = a^2(1 - e^2)$

$\Rightarrow x^2(e^2 - 1) - y^2 = a^2(e^2 - 1)$

$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$

Put $a^2(e^2 - 1) = b^2$

$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the required eqn.

Note –1) The line AA_1 is called tranverse axis.

2) If B & B_1 be points on OY such that $OB = OB_1 = b$, then BB_1 is called conjugate axis.

3) the chord passing through the focus paralled to directrix is called latus rectum and length of latus rectum is $2b^2/a^2$ or $2a^2(e^2-1)/a^2$ or $2(e^2-1)$

5.5. FOCAL DISTANCES OF A POINT ON THE HYPERBOLA

Let $P(x, y)$ be any point on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{[Fig. Art. 5.2]}$$

and S, S_1 be its foci.

If PMM_1 be $\perp$ to the directrices, we have

$$SP = ePM = e(x - CZ) = e(x - a/e) = ex - a$$

and $$S_1P = ePM_1 = e(x + CZ_1) = e(x + a/e) = ex + a$$

Therefore $$S_1P - SP = 2a.$$

Hence the difference of the focal distances of any point on the hyperbola is constant and is equal to the transverse axis.

From the above it follows that *the hyperbola is the locus of a point which moves such that the difference of distances from two fixed points is constant.*

5.6. PARAMETRIC EQUATION TO THE HYPERBOLA

Plainly the point $x = a\sec\theta$ and $y = b\tan\theta$ always satisfies the equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and is therefore a point on the hyperbola whatever value θ may have.

Hence for all values of θ, the equations

$$\boldsymbol{x = a\sec\theta,\ y = b\tan\theta}$$

represent points on the hyperbola.

The co-ordinates of any point on the hyperbola can also be expressed as

$$x = a\cosh t \text{ and } y = b\sinh t.$$

5.7. PARTICULAR CASES

The equation of the hyperbola in standard form is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Replacing b by a, the above equation reduces to

$$\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1 \text{ which } \Rightarrow x^2 - y^2 = a^2.$$

This equation defines the equation of a rectangular hyperbola. Thus the equation of a *rectangular hyperbola* is $\boldsymbol{x^2 - y^2 = a^2}$.